DECAY OF LONGITUDINAL PLASMA OSCILLATIONS TO ION-SOUND OSCILLATIONS

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Recently, the question of the nonlinear relation between various plasma oscillations has been the subject of much attention as a result of a series of circumstances. The most important of these is the fact that in the majority of experiments on beam instabilities [1, 2] the intensity of the oscillations excited is very large, so that nonlinear effects in the interaction of oscillations must be significant. It should be noted that beam instability is not the only method of exciting high-frequency plasma oscillations. As was shown in [3], very intense oscillations may also be excited by beams of transverse waves of various frequency ranges, among which are powerful light beams [4]. Finally, excitation is possible by means of shock waves [5] and large-amplitude waves propagating through a plasma.

Nonlinear coupling of plasma and low-frequency ion-sound oscillations leads, in particular, to the generation of the latter [6]. On the one hand, this is of interest as regards the problem of turbulent heating of a plasma, since the absorption of ion-sound oscillations in a plasma is usually stronger than the absorption of plasma oscillations. On the other hand, ion-sound oscillations may bring about the acceleration of low-energy ions due to the effects of induced Čerenkov absorption and radiation of waves by ions, as considered in the work of one of the authors [7]. Although plasma oscillations accelerate particles more effectively [8], the injection conditions in the configuration for acceleration by plasma oscillations are very stringent $v > v_e$. The number of ions with such velocity for small ion temperatures T_i is small. Thus, the acceleration of ions will arise in this case as a result of the interaction of ion-sound oscillations until such time as their velocity reaches values of the order v_e. This question is of interest not only for the acceleration of ions (heating) in the presence of high-frequency turbulence created by beams of charged particles or as a result of the action of powerful radiation on a plasma, but also for the problem of neutron radiation from powerful impulse discharges in a plasma and for a series of astrophysical problems.

In what follows we consider a number of one-dimensional self-consistent problems regarding the interaction (decay and fusion) of plasma and ion-sound oscillations resulting from the induced Raman scattering of the former by the latter. It is shown that the development of instability in a turbulent plasma with a high level of excited plasma oscillations leads both to the excitation of ion-sound oscillations, and also to the appearance in the plasma oscillation spectrum of satellites differing from the basic frequency ω_{0e} by a frequency of the order ω_{0i} and with greater intensities for the lower frequencies. The qualitative change of the plasma oscillation spectrum may serve as an immediate indication of the excitation of ion-sound oscillations in the system. The results obtained allow one to trace the process of development of instabilities. It is shown that in a plasma with a high level of ion-sound oscillations "violet" satellites are excited in the plasma oscillation spectrum, while the intensities of the violet satellites have a tendency to level out and form a satellite plateau if the level of ion-sound waves is high enough.

1. High-intensity plasma waves propagating in a plasma become unstable with respect to the excitation of ion-sound waves. The instability mechanism corresponds to the effects of induced decay and fusion of waves. By k_1 we shall designate the original wave number of the plasma waves, and by k_2 and k_3 the wave numbers of the plasma and ion-sound waves into which the original wave decays. The conservation laws for decay have the form

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_s, \qquad \omega_{01} = \omega_{02} + \omega_s. \tag{1.1}$$

We shall make use of the dispersion equation for plasma and ion-sound waves

$$\omega_{01}^2 = \omega_{0e}^2 + 3v_e^2 k_1^2, \quad \omega_{0e}^2 = 4\pi \frac{e^2}{m_e} n_0, \quad v_e = \left(\frac{T_e}{m_e}\right)^{1/2},$$
$$\omega_s^2 = \frac{k_s^2 v_s^2}{1+\lambda_e^2 k_s^2}, \quad v_s = \left(\frac{T_e}{m_i}\right)^{1/2} = v_e\left(\frac{m_e}{m_i}\right)^{1/2}, \quad \lambda_e = \frac{v_e}{\omega_0}. \tag{1.2}$$

Then for

$$|\mathbf{k}_1| \ll 1/\lambda_e, \qquad |\mathbf{k}_s| = |\mathbf{k}_1 - \mathbf{k}_2| < 1/\lambda_e,$$

the second equation of (1.1) may be represented in the form

$$\frac{3}{2}\frac{v_e^2}{\omega_0}(k_1^2 - k_2^2) = |\mathbf{k}_s| v_e\left(\frac{m_e}{m_i}\right)^{1/2}\left(1 - \frac{1}{2}\lambda_e^2 k_s^2\right). \tag{1.3}$$

We shall analyze the conservation laws (1.1) and (1.3) for the one-dimensional case, first assuming that k_1, k_2, k_s are collinear. To be specific, let $k_1 > 0$. Then it is clear from (1.3) that $k_1 > |k_2|$; consequently, $k_1 - k_2 = k_s > 0$ the sign of the modulus on the right side of (1.3) may be omitted; shortening the equation to k_s and using (1.1), we obtain

$$k_s = 2(k_1 - k_0), \qquad k_0 = \frac{1}{3\lambda_e}\left(\frac{m_e}{m_i}\right)^{1/2}. \tag{1.4}$$

In Eq. (1.4) we have neglected the term

$$k_0 \lambda_e^2 k_s^2 = k_s\left(\frac{m_e}{m_i}\right)^{1/2}\frac{\lambda_e k_s}{3}$$

in comparison* with k_s. The wave number $k_s > 0$; thus it follows from (1.4) that decay processes are possible in the one-dimensional case only for $k_1 > k_0$.

Thus, if there is an initial narrow $\Delta k_1 < (2k_0/k)\cdot(k_1 - k_0)$ spectrum in the plasma with number of waves $N^l(k_1)$, where $k_1 > 0$, then decays are possible in the one-dimensional case for which an ion-sound spectrum with a number of waves $N^s(k_s)$ and a spectrum-satellite of plasma waves $N^l(k_2)$, where

$$k_s = 2(k_1 - k_0), \qquad k_2 = k_1 - k_s = 2k_0 - k_1. \tag{1.5}$$

Moreover, if the width of the spectrum $N^l(k_1)$ is equal to Δk_1, then the spectral widths $N^l(k_2)$ and $N^l(k_s)$ will be equal to $\Delta k_2 = \Delta k_1$, $\Delta k_s = 2\Delta k_1$ respectively. Since $k_1 > k_0$, it is clear from (1.5) that k_2 lies in the interval

$$-1/\lambda_e \ll k_2 < k_0.$$

*For calculating $\omega_s \sim \omega_{0i}$ with $k_s \sim 2k_1 \lesssim 1/\lambda_e$ at the limit of applicability of (1.3) this term is significant.

If the resulting $|k_2| < k_0$, then further decays and the appearance of satellites with a lower frequency are forbidden. Generalizing from this, we easily conclude that if the initial k_1 lies within the limits $k_0 < k_1 < 3k_0$, then only one satellite may appear. For $3k_0 < k_1 < 5k_0$ there may be two satellites, for $5k_0 < k_1 < 7k_0$ — three satellites, etc., and for $(2n-1)\,k_0 < k_1 < (2n+1)\,k_0$ n red satellites may appear.

If, in addition to the given initial narrow spectrum with wave number k_1, there is a broad spectrum of ion-sound noises in the plasma with a sufficiently high level, fusion processes are possible, leading to the appearance of violet satellites with $\mathbf{k}_2 = \mathbf{k}_1 + \mathbf{k}_s$, where $|\mathbf{k}_2| > |\mathbf{k}_1|$ and, consequently, $\omega_2 > \omega_1$.

If we carry out the change of indices $1 \rightleftarrows 2$ the analysis of the conservation laws remains the same as before.

For the three-dimensional treatment we obtain instead of (1.4)

$$|\mathbf{k}_s| = 2\,(k_1 \cos\theta - k_0), \quad k_1 = |\mathbf{k}_1|. \qquad (1.6)$$

if $\mathbf{k}_s$ is directed at an angle θ to $\mathbf{k}_1$, i.e., decays are allowed when

$$k_1 > k_0, \qquad \cos\theta > \cos\theta_{max} = \frac{k_0}{k_1}. \qquad (1.7)$$

The allowed angle is

$$\theta < \theta_{max} = {}^1/_2\pi - \delta \text{ for } |\mathbf{k}_1| \leqslant {}^1/_2\lambda_e, \quad \delta = {}^2/_3\,(m_e / m_i)^{1/2}.$$

For $|\mathbf{k}_1| \to k_0$ decays are allowed for $\theta \to 0$, i.e., one-dimensionality is automatically ensured. From (1.6) it is clear that for a given $\mathbf{k}_1$ decays are possible with $0 < |\mathbf{k}_s(\theta)| < 2\,(k_1 - k_0)\,|$, while $0 < \theta < \theta_{max}$. For given k_1 and θ one may easily obtain

$$|\mathbf{k}_2(\theta)| = \sqrt{(|\mathbf{k}_1| - k_0)^2 + 4k_0|\mathbf{k}_1|(1-\cos\theta)}. \qquad (1.8)$$

i.e., $|\mathbf{k}_2(\theta)|$ are permitted within the limits

$$|\mathbf{k}_2(\theta_{max})| \geqslant |\mathbf{k}_2(\theta)| \geqslant |\mathbf{k}_2(0)|. \qquad (1.9)$$

2. We shall now find $w^{sll}(K_s, K_1, K_2)$ the probability of induced absorption of the waves $K_1 = \{\mathbf{k}_1, \omega_1\}$ and $K_2 = \{\mathbf{k}_2, \omega_2\}$ and of radiation of the wave $K_s = \{\mathbf{k}_s, \omega_s\}$. Within the limits of small energy values of ion-sound waves the expression for the rate of change of this energy in connection with plasma wave decays has the form

$$\frac{\partial W^s}{\partial t} = \frac{\partial}{\partial t}\int \frac{\omega_s N^s(\mathbf{k}_s)\,d\mathbf{k}_s}{(2\pi)^3} = \int \frac{d\mathbf{k}_s\omega_s}{(2\pi)^3}\Big[\int N_1{}^l(\mathbf{k}_1)\,N_2{}^l(\mathbf{k}_2)\times$$
$$\times\, w^{sll}(K_s, K_1 - K_2) - w^{sll}(K_s, -K_1, K_2)\Big]\,d\mathbf{k}_1 d\mathbf{k}_2,$$
$$\frac{N(\mathbf{k})}{(2\pi)^3} = \frac{1}{4\pi} E_\mathbf{k}{}^2 \frac{\partial\varepsilon(\omega_1\mathbf{k})}{\partial\omega}. \qquad (2.1)$$

This expression was compared with the expression obtained in the second field approximation on solving the kinetic equation for the energy conversion of ion-sound waves due to nonlinear interaction between the plasma waves

$$\frac{\partial W^s}{\partial t} = \frac{1}{4\pi}\left\langle \mathbf{E}_s{}^{(2)} \frac{\partial \mathbf{D}_s{}^{(2)}}{\partial t}\right\rangle. \qquad (2.2)$$

Here the Fourier component

$$\mathbf{E}_{s\mathbf{k},\omega}^{(2)} = \frac{4\pi i}{\omega\varepsilon^l(\omega,\mathbf{k})}\, j_{\mathbf{k},\omega}^{(2)}, j_{\mathbf{k},\omega} = \frac{e\omega_0{}^2\mathbf{k}\omega}{8\pi m_e k^2 v_e{}^2}\times$$
$$\times \int (\mathbf{E}_{k_1\omega_1}\mathbf{E}_{k_2\omega_2})\,\frac{1}{\omega_1\omega_2}\,\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})\,\delta(\omega_1 + \omega_2 - \omega)\times$$
$$\times\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\omega_1\, d\omega_2.$$

Here $N^l(\mathbf{k}_1)$, $N^l(\mathbf{k}_2)$ and, $N^s(\mathbf{k}_s)$ are the numbers of plasma and ion-sound waves (with wave numbers $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_s$, respectively) arriving in unit volume of plasma.

One may easily obtain the expression for the probability of the decay process $\mathbf{k}_1 \to \mathbf{k}_2 + \mathbf{k}_s$ by comparing (2.1) and (2.2) (taking into consideration that $\omega = \omega(\mathbf{k})$):

$$w^{sll}(K_s, K_1, -K_2) = \frac{e^2\omega_s{}^3 m_i}{64\pi m_e{}^3 v_e{}^4 k_s{}^2}\,\frac{(\mathbf{k}_1\mathbf{k}_2)^2}{k_1{}^2 k_2{}^2}\,\delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_s)\,\delta\times$$
$$\times\left(\frac{3}{2}\frac{v_e{}^2}{\omega_0}(k_1{}^2 - k_2{}^2) - |\mathbf{k}_s|\,v_s\right). \qquad (2.3)$$

Here the δ function factors correspond to the conservation of energy and momentum in decays. We note that certain qualitative estimates and cross sections of decay processes were obtained in [9], and that the latter differ from (2.3) by the factor ${}^1/_2 k_s{}^2\lambda_e{}^2 = (\omega\partial\varepsilon / \partial\omega)^{-1}$.

We shall estimate the increment of plasma wave instability for decay to ion-sound and plasma waves with lower frequencies and wave numbers.

For an order of magnitude estimate of the increment of ion-sound wave generation, for example, we may use the expression

$$\gamma^s(\mathbf{k}_s) \approx \frac{1}{N^s(\mathbf{k}_s)}\,\frac{\partial N^s(\mathbf{k}_s)}{\partial t} =$$
$$= \int N^l(\mathbf{k}_1)\, w^{sll}(K_s, K_1, -K_2)\, d\mathbf{k}_1\, d\mathbf{k}_2. \qquad (2.4)$$

Setting the expression for w^{sll} in (2.4), and taking the spectrum $N^l(\mathbf{k}_1)$ to be one-dimensional and fairly narrow we obtain

$$\gamma^s(\mathbf{k}_s, \mathbf{k}_1) =$$
$$= \frac{\pi}{48}\,\frac{e^2\omega_0}{m_e{}^2 v_e{}^3}\left(\frac{m_e}{m_i}\right)^{1/2} N^l(\mathbf{k}_1)\left[\frac{1}{\cos\theta}\,\frac{(k_1 - k_s\cos\theta)^2}{(k_1 - k_s\cos\theta)^2 + k_s{}^2\sin^2\theta}\right],$$
$$k_s = 2\,(k_1\cos\theta - k_0),$$
$$\cos\theta = \frac{\mathbf{k}_1\mathbf{k}_s}{k_1 k_s} \geqslant \frac{k_0}{k_1},\ k_1 = |\mathbf{k}_1|,\ k_s = |\mathbf{k}_s|. \qquad (2.5)$$

For a given $\mathbf{k}_1$ the expression in brackets had a maximum for $\cos\theta_1 = k_0 / k_1$ and the increment of decay instability is in order of magnitude somewhat less for the one-dimensional (by a factor of k_1/k_0) than for the three-dimensional case*:

$$\gamma_{max}^{sll} \approx \frac{\pi}{16}\,\frac{e^2\omega_0}{m_e{}^2 v_e{}^3}\left(\frac{m_e}{m_i}\right)^{1/2} N^l(\mathbf{k}_1)\,\frac{k_1}{k_0} = \frac{3}{64}\,\omega_0\,\frac{k_0}{\Delta k_0}\,\frac{W}{n m_e v_e{}^2}, \qquad (2.6)$$

where W^l in the total energy of the spectrum $N^l(\mathbf{k}_1)$.

We shall now compare the decay instability increment with the increment characteristic of the transfer

*It must be kept in mind that the maximal increment corresponds to $|\mathbf{k}_s| = 0$, i.e., virtually to the absence of decay.

of energy over the spectrum of non-one-dimensional longitudinal waves (see Eq. (12) [10])

$$\gamma_{(1)}^{ll} \approx \frac{\omega_0}{(2\pi)^{5/2}} \frac{W^l}{m_e n_0 v_e^2} \left(\frac{v_e}{v_\varphi^l}\right)^3, \qquad v_\varphi^l = \frac{\omega_0}{|\mathbf{k}_1|}. \quad (2.7)$$

We finally obtain

$$\frac{\gamma_{(1)}^{ll}}{\gamma_{\max}^{sll}} = \frac{64}{3(2\pi)^{5/2}} \frac{\Delta k}{k_1} (\lambda_e k_1)^3. \quad (2.8)$$

It can easily be seen that

$$\gamma_{(1)}^{ll} / \gamma_{\max}^{sll} \ll 1 \quad \text{for } k_1 \ll 1/\lambda_e, \ \Delta k / k_1 \ll 1.$$

In the one-dimensional case

$$\gamma_{(2)}^{ll} = \gamma_{(1)}^{ll} (k_1 \lambda_e)^2.$$

Consequently,

$$\gamma_{(2)}^{ll} / \gamma^{sll} = \frac{64}{(2\pi)^{5/2}} (\lambda_e k_1)^6 \frac{\Delta k_1}{k_1} \left(\frac{m_i}{m_e}\right)^{1/2} \ll 1. \quad (2.9)$$

Thus the processes under consideration proceed more rapidly than the processes of spectral transfer of longitudinal plasma waves.

The mutual interaction of generated acoustic oscillations will also be of little significance. Thus estimates show that for acoustic oscillation energy such as is attained as a result of the development of a first satellite of the order $W^s \sim W^l \omega_s/\omega_0$ (see ¶6), the characteristic ion-sound wave interaction time is greater by a factor of

$$(m_i / m_e)(k / \Delta k)(1 / k\lambda_e).$$

than the characteristic time for development of the instability under consideration.

3. The basic equations describing oscillation interactions as a result of decay processes may be found by considering all possible induced transitions arising in the plasma due to the presence of decays described by the probability found above, and also of their reverse processes, whose probabilities may easily be found from the principle of balance [11]

$$\frac{\partial N^s(\mathbf{k}_s)}{\partial t} = \int N^l(\mathbf{k}_1) N^l(\mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2 \,[w^{sll}(K_s, -K_1, K_2) + w^{sll}(K_s, K_1, -K_2)] + N^s(\mathbf{k}_s) \int N^l(\mathbf{k}_1)\, d\mathbf{k}_1 \times$$
$$\times \int d\mathbf{k}_2\, [w^{sll}(K_s, K_1, -K_2) - w^{sll}(K_s, -K_1, K_2)] + N^s(\mathbf{k}_s) \int N^l(\mathbf{k}_2)\, d\mathbf{k}_1 \times$$
$$\times \int d\mathbf{k}_1\, [w^{sll}(K_s, -K_1, K_2) - w^{sll}(K_s, K_1, -K_2)],$$

$$\frac{\partial N^l(\mathbf{k}_1)}{\partial t} = \int N^s(\mathbf{k}_s) N^l(\mathbf{k}_2)\, d\mathbf{k}_s d\mathbf{k}_2\, [w^{sll}(K_s, K_1, -K_2) + w^{sll}(K_s, -K_1, K_2)] + N^l(\mathbf{k}_1) \int N^s(\mathbf{k}_s)\, d\mathbf{k}_s \times$$
$$\times \int d\mathbf{k}_2\, [-w^{sll}(K_s, K_1, -K_2) - w^{sll}(K_s, -K_1, K_2)] + N^l(\mathbf{k}_1) \int N^l(\mathbf{k}_2)\, d\mathbf{k}_2 \times \quad (3.1)$$
$$\times \int d\mathbf{k}_s\, [w^{sll}(K_s, -K_1, K_2) - w^{sll}(K_s, K_1, -K_2)],$$

$$\frac{\partial N^l(\mathbf{k}_2)}{\partial t} = \int N^s(\mathbf{k}_s) N^l(\mathbf{k}_1)\, d\mathbf{k}_1 d\mathbf{k}_s\, [w^{sll}(K_s, -K_1, K_2) + w^{sll}(K_s, K_1, -K_1)] + N^l(\mathbf{k}_2) \int N^s(\mathbf{k}_s)\, d\mathbf{k}_s \times$$
$$\times \int d\mathbf{k}_1\, [-w^{sll}(K_s, -K_1, K_2) - w^{sll}(K_s, K_1, -K_2)] + N^l(\mathbf{k}_2) \int N^l(\mathbf{k}_1)\, d\mathbf{k}_1 \times$$
$$\times \int d\mathbf{k}_s\, [w^{sll}(K_s, K_1, -K_2) - w^{sll}(K_s, -K_1, K_2)].$$

The law of conservation has the form

$$\frac{d}{dt}\left[\int N^l(\mathbf{k}_1)\, \omega_{01}\, d\mathbf{k}_1 + \int N^l(\mathbf{k}_2)\, \omega_{02}\, d\mathbf{k}_2 + \int N^s(\mathbf{k}_s)\, \omega_s\, d\mathbf{k}_s\right] = 0 \quad (3.2)$$

We shall introduce the one-dimensional distribution functions

$$\int N^l(\mathbf{k}_1)\, d\mathbf{k}_{1\perp} = (2\pi)^2 \sum_{\mathbf{k}_{1\perp}} N^l(\mathbf{k}_1) = (2\pi)^2 N(k_1),$$

($N^l(k_2)$, $N_s(k^s)$, by analogy).

We shall confine ourselves to the one-dimensional process, which will develop if the ion-sound and plasma noises necessary to "seed" the process and cause the system to pass to states of unstable equilibrium have a one-dimensional character.

We shall dwell on the question of the nonlinear coupling of two narrow spectra. Integrating system (3.1) with account for the expression for the probability (2.3), we can reduce the problem to a system of first-order equations

$$\frac{\partial N_1^l}{\partial t} = \alpha(N_2^l N^s - N_1^l N^s - N_1^l N_2^l), \qquad N^l(k_1) \equiv N_1^l$$
$$\frac{\partial N_2^l}{\partial t} = -\alpha(N_2^l N^s - N_1^l N^s - N_1^l N_2^l), \qquad N^l(k_2) \equiv N_2^l$$
$$\frac{\partial N^s}{\partial t} = -\frac{\alpha}{2}(N_2^l N^s - N_1^l N^s - N_1^l N_1^l), \qquad N^s(k_s) \equiv N^s$$
$$\alpha = \frac{\pi e^2}{24 m_e^2 v_e^2 \lambda_e} \left(\frac{m_e}{m_i}\right)^{1/2} = \frac{\omega_0^2 k_0}{32 n_0 m_e v_e^2}. \quad (3.3)$$

It may easily be verified that the conservation law is fulfilled:

$$\frac{d}{dt}[N_1^l \omega_{01} \Delta k_1 + N_2^l \omega_{02} \Delta k_2 + N^s \omega_s \Delta k_s] = 0. \quad (3.4)$$

System (3.3) has the integrals

$$N_1^l + N_2^l = A, \qquad N_1^l + 2N^s = B. \quad (3.5)$$

4. We shall consider the question of the generation by a narrow spectrum of plasma waves N_1^l of a spectrum of acoustic waves N^s and of a single red spectral-satellite N_2^l c $|k_2| < |k_1|$, which will be realized if, for example, at the initial moment t = 0 there is no ion-sound wave field $N^{s0} = 0$ and a small number of "seeding" quanta $N_2^{l0} \ll N_1^{l0}$ are present. Substituting (3.5) in (3.3) we get, assuming $A = N_1^{l0} + N_2^{l0}$, $B = N_1^{l0}$,

$$\frac{\partial N_1^l}{\partial t} = -\frac{\alpha}{2}(4N_1^l - N_1^{l0})\left(N_1^{l0} - N_1^l + \frac{2}{3} N_2^{l0}\right). \quad (4.1)$$

Everywhere in the calculations we have neglected N_2^{l0} in comparison with N_1^{l0}, with the exception of the term in the second bracket, since it is indispensable for the system to pass from the state of unstable equilibrium. It is clear from expression (4.1) that as a result of the development of an instability the system of two lines will pass to a state with $N_1^{l\infty} = {}^1/_4 N_1^{l0}$, $N_2^{l\infty} = {}^3/_4 N_1^{l0}$, $N^{s\infty} = {}^3/_8 N_1^{l0}$

The solution of (4.1) has the form

$$N_1^l(t) = N_1^{l0} \frac{N_2^{l0} + {}^9/_2 N_1^{l0} \exp(-{}^3/_2 N_1^{l0}\alpha t)}{4N_2^{l0} + {}^9/_2 N_1^{l0} \exp(-{}^3/_2 N_1^{l0}\alpha t)}. \qquad (4.2)$$

We note that the process of spectral redistribution of waves does not depend on k_1 for $k_1 \ll 1/2\lambda_e$ for $k_1 \sim 1/2\lambda_e$ the increment α changes to $\alpha(1 + 4k_1^2\lambda_c^2)$, i.e., the transfer slows down. A consideration of the process of the formation of a single red satellite is justified by the fact that such a process is an "elementary cell" of the complicated process of the evolution of any broad one-dimensional spectrum of plasma waves in the presence of a sufficiently low noise level.

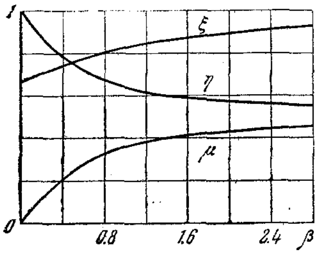

Fig. 1

5. We shall consider the problem of the growth of a single violet satellite N_1^l of a given narrow basic spectrum, designated by N_2^l, when there is a high level of ion-sound waves in the plasma. Without restricting the generality of the argument, we may assume that k_2 lies in the limits $1/\lambda_e \ll k_2 < k_0$, and that the wave number of the violet satellite $k_1 > k_0$. We may then use the same system (3.3), while for the appearance and growth of a violet satellite with $k_1 = 2k_0 - k_1$ we must have present in the spectrum $N^s(k_s)$ waves with

$$k_s = k_1 - k_2 = 2(k_0 - k_2) > 0.$$

We note that for $|k_2| < k_0$ two violet satellites (on fusion with ion-sound waves) with directions parallel and antiparallel to that of k_2 may be formed. Determining the constants A and B in the integrals (3.5) in accordance with the problem under consideration

$$N_1^l + N_2^l = N_2^{l0}, \qquad N_1^l + 2N^s = 2N^{s0}. \qquad (5.1)$$

we obtain a further equation for N_2^l, setting (5.1) in (3.3),

$$\frac{\partial N_2^l}{\partial t} = -\frac{\alpha}{2}[4(N_2^l)^2 + N_2^l(4N^{s0} - 5N_2^{l0}) + N_2^{l0}(N_2^{l0} - 2N^{s0})]. \qquad (5.2)$$

For t = 0 the expression in square brackets is positive, and so N_2^l begins to decrease to the value $N_2^{l\infty}$. The equation for $\eta = N_2^{l\infty}/N_2^{l0}$, corresponding to a zero on the right side of (5.2), will then assume the form

$$\eta^2 + \eta(\beta - {}^5/_4) + {}^1/_4(1 - 2\beta) = 0, \qquad \beta = N^{s0}/N_2^{l0}. \quad (5.3)$$

The solution is

$$\eta = {}^1/_8[5 - 4\beta + \sqrt{(5 - 4\beta)^2 + 16(2\beta - 1)}]. \qquad (5.4)$$

The root is taken with a plus sign outside the radical, since the ratio N_2^{l0}/N_2^{l0}, becoming less than 1, attains a stable value η, and for $N_2^l / N_2^{l0} < \eta$ the increment changes sign and N_2^l does not suffer any further change; however, the other root of (2.4) is always less than η. Figure 1 illustrates the relationships

$$\eta = \eta(\beta), \qquad \mu(\beta) = \frac{N_1^{l\infty}}{N_2^{l0}} = 1 - \eta(\beta), \qquad \xi(\beta) = \frac{N^{s\infty}}{N^{s0}} = 1 - \frac{\mu(\beta)}{2\beta}.$$

For $N^{s0} \gg N_2^{l0}$ almost half a quantum of the original spectrum passes to the violet satellite $\eta = N_2^{l\infty}/N_2^{l0} \to 0.5$, i.e., an evening out of the spectra takes place. If the spectrum N^S is broad enough and the process of formation of a second violet satellite proceeds further, then there is clearly a tendency for them to level out with regard to numbers of waves and to form a "satellite plateau." Since $\mu(\beta) \to {}^2/_3$ for $\beta \to 0$, for sufficiently small N^{S0} only a third of the ion-sound waves takes part in fusion with plasma waves (and for large N^{S0} an even smaller portion). For $N^{s0} \gg N_2^{l0}$ one may practically assume that N^S is independent of time.

In conclusion, we may mention that sound waves responsible for the appearance of violet satellites may not automatically be formed on the decay of N_2 with the formation of a red satellite, since these waves correspond to various directions k_S. We also note that if $|k_2| > k_0$, then, if a given corresponding N^{S0} is present at the initial moment, we may restrict ourselves to considering one violet satellite only, without the simultaneous formation of red ones, only in order to clarify the nature of the basic processes involved in a complex system of lines whose evolution is in reality described by a system of coupled equations.

6. We shall now consider the process of evolution of three lines: the basic line given at the initial moment N_1^l, and the two corresponding to the red satellites N_2^l and N_3^l.

For $k_1 > 3k_0$ in accordance with the laws of conservation, such a process may be realized. The wave number corresponding to the first satellite is $k_2 = 2k_0 - k_1$, and to the second $k_3 = k_1 - 4k_0$.

We note that for a one-dimensional treatment the spectra N_1^S and N_2^S associated with the transitions $N_1^l \rightleftarrows N_2^l$ and $N_2^l \rightleftarrows N_3^l$, do not overlap, since k_{S1} and k_{S2} are of different signs.

Analysis also shows that the waves of the spectrum N_1^S cannot result in the appearance of yet another line, distinct from the three under consideration, by fusion with waves from N_3^l, even for a wide initial spectrum N_1^l.

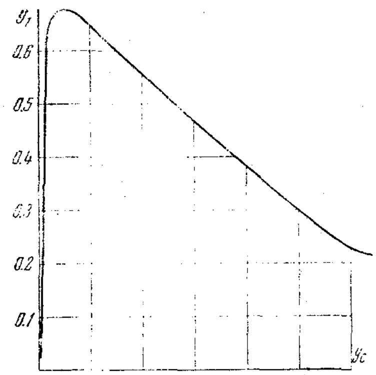

Fig. 2

In the three-dimensional case N_1^S and N_2^S may overlap for specific angles, and the picture will be more complicated. The system of equations for a process with two red satellites N_2^l, N_3^l and a given basic line N_1^l has the form

$$\frac{\partial N_1^l}{\partial t} = \alpha(N_2^l N_1^s - N_1^l N_1^s - N_1^l N_2^l), \quad \frac{\partial N_2^l}{\partial t} =$$

$$= -\alpha(N_2^l N_1^s - N_1^l N_1^s - N_1^l N_2^l) + \alpha(N_3^l N_2^s - N_2^l N_2^s - N_2^l N_3^l),$$

$$\frac{\partial N_3^l}{\partial t} = -\alpha(N_3^l N_2^s - N_2^l N_2^s - N_2^l N_3^l), \quad \frac{\partial N_1^s}{\partial t} =$$

$$= -\frac{\alpha}{2}(N_2^l N_1^s - N_1^l N_1^s - N_1^l N_2^l),$$

$$\frac{\partial N_2^s}{\partial t} = -\frac{\alpha}{2}(N_3^l N_2^s - N_2^l N_2^s - N_2^l N_3^l). \qquad (6.1)$$

The system (6.1) has three integrals

$$N_1^l + N_2^l + N_3^l = \mathrm{const}_1 = N_1^{l0}, \qquad (6.2)$$
$$N_1^l + 2N_2^s = \mathrm{const}_2 = N_1^{l0} + 2N_1^{s0},\ N_3^l - 2N_2^s = \mathrm{const}_3 = -2N_2^{s0}.$$

We shall assume that

$$N_2^{s0} = N_1^{s0} = N^{s0} \ll N_1^{l0} \quad (\text{for } t = 0)$$

(it is necessary, however, that for the process to start $N^{s0} \neq 0$, and take the quantities N_2^{l0} and N_3^{l0} equal to zero. Using (6.2), we obtain the system

$$\frac{\partial N_2^l}{\partial t} = \frac{\alpha}{2}\left[-4N_2^l(N_2^l + 2N_3^l) + N_1^{l0}(N_3^l + 3N_2^l) + 2N_1^{l0}N^{s0}\right],$$
$$\frac{\partial N_3^l}{\partial t} = \frac{\alpha}{2}\left[N_3^l(3N_2^l - N_3^l) + 2N^{s0}(N_2^l - N_1^l)\right]. \qquad (6.3)$$

Setting the square brackets equal to zero, we obtain that in the final state $N_1^{l\infty} = {}^2/_{14}N_1^{l0}$, $N_2^{l\infty} = {}^3/_{14}N_1^{l0}$, $N_3^{l\infty} = {}^9/_{14}N_1^{l0}$, $N_1^{s\infty} = {}^6/_{14}N_1^{l0}$, $N_2^{s\infty} = {}^9/_{28}N_1^{l0}$ i.e., there is a tendency for any spectrum of longitudinal waves to "creep" into the red region.

Eliminating time in (6.3), we obtain an equation for sufficiently pronounced $y_1 = N_2^l / N_1^{l0}$ and $y_2 = N_3^l / N_1^{l0}$ (i.e., neglecting N^{s0} everywhere)

$$\frac{dy_1}{dy_2} = \frac{y_1(3 - 4y_1) + y_2(1 - 8y_1)}{y_2(3y_1 - y_2)}. \qquad (6.4)$$

An approximate solution of this equation has the form (Fig. 2)

$$y_1 = 0.750 - 0.742\left(\frac{0.004}{y_2}\right)^{1/3} - 0.952\,y_2 + \frac{0.06\cdot 0.074}{0.702 - y_1}. \qquad (6.5)$$

Here the initial point is chosen so that $y_1^0 = y_2^0 = 0.004$.

We may conclude from the form of the function $y_1(y_2)$ that the first satellite increases sharply, at first attaining the value $y_{1_{\max}} = N_2^{l\,\max}/N_1^{l0} = 0.68$ for $y_2 = 0.043$. At this stage the increase of the second is vanishingly small; subsequently, N_2^l begins to decrease in connection with the growth of the second satellite, which becomes the most intense.

7. We shall conclude with the case when the initial spectrum $N^l(k_1)$ corresponds to $k_1 \gg 4k_0$ and the width $\Delta k_1 \gg 4k_0$. Mathematically, the problem is fairly complicated; we may base a qualitative discussion on the results already obtained.

Waves of any small portion Δk_1 of the spectrum $N_1^{l+}(k_1)$, corresponding, for example, to $k_1 > 0$, will create a satellite at the portion Δk_2 with $k_2 = 2k_0 - k_1 < 0$, and this ensures the increase of the number of waves at the portion close to $k_1 - 4k_0$. Since $\Delta k_1 \gg 4k_0$, we get the following picture: the spectrum N_1^{l+} will create a spectral-satellite N_2^{l-} of the same width with wave numbers of opposite sign. Both spectra will spread, being displaced to the region of wave numbers of smaller modulus (and frequency). In this process individual waves will pass many times from N_1^{l+} to N_2^{l-} and back again, decaying with each transition to an ion-sound wave and a plasma wave (of lower frequency and opposite direction), gradually losing their energy in the excitation of ion-sound waves and approaching the value of the wave number k_0.

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